DISTRIBUTIVE LATTICES OF TILTING MODULES AND SUPPORT τ-TILTING MODULES OVER PATH ALGEBRAS

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Abstract. In this paper we study the poset of basic tilting kQ-modules when Q is a Dynkin quiver, and the poset of basic support τ-tilting kQ-modules when Q is a connected acyclic quiver respectively. It is shown that the first poset is a distributive lattice if and only if Q is of types $\mathbb{A}_1, \mathbb{A}_2$ or $\mathbb{A}_3$ with a nonlinear orientation and the second poset is a distributive lattice if and only if Q is of type $\mathbb{A}_1$.

1. Introduction

Let Q be a finite connected acyclic quiver and kQ be the path algebra of Q over an algebraically closed field k. Denote by mod-kQ the category of finite dimensional right kQ-modules, by ind-kQ the category of indecomposable modules in mod-kQ and by Γ(mod kQ) the Auslander-Reiten quiver of kQ. For $M \in$ mod-kQ, we denote by add M (respectively, Fac M, Sub M) the category of all direct summands (respectively, factor modules, submodules) of finite direct sums of copies of M and by $|M|$ the number of pairwise non-isomorphic indecomposable direct summands of M. Let P_i be an indecomposable projective module in mod-kQ associated with vertex $i \in Q_0$ and τ be the Auslander-Reiten translation.

Tilting theory for kQ, or more generally for a finite dimensional basic k-algebra, was first appeared in [3] and have been central in the representation theory of finite dimensional algebras since the early seventies. For the classical tilting modules and their mutation theory, there is a naturally associated quiver named tilting quiver which is defined in [13]. Happel and Unger defined a partial order on the set of basic tilting modules and showed that the tilting quiver coincides with the Hasse quiver of this poset [4]. A related partial order has been studied in the τ-tilting theory introduced in [2] and the analog result also holds, that is, the support τ-tilting quiver also coincide with the Hasse quiver of this related partial order.

Recently, the lattice structure of the poset of tilting modules and support τ-tilting modules have been studied in [6, 7, 12]. More precisely, Kase showed that for representation-infinite algebras kQ, the poset of its pre-projective tilting modules possess a distributive lattice structure if and only if the degree of all vertices in Q are greater than 1 [7]. Later Iyama, Reiten, Thomas and Todorov proved that for path algebras kQ, the poset of its support τ-tilting modules possess a lattice structure if and only if Q is a Dynkin quiver or has at most 2 vertices.

The aim of this paper is to study the following problem.

Problem 1.1. *Let Q be a finite connected acyclic quiver.*

(1) When does the poset of basic tilting kQ-modules possess a distributive lattice structure?

(2) When does the poset of basic support τ-tilting kQ-modules possess a distributive lattice structure?

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Our main result is the following theorem.

Theorem 1.2. *Let Q be a Dynkin quiver. Then the following statements are equivalent.*
(1) All tilting modules are slice modules.
(2) The full subquiver generated by any tilting module form a section of $\Gamma(\mathrm{mod}\ kQ)$.
(3) The tilting quiver $\vec{\mathcal{T}}(Q)$ is a distributive lattice.
(4) Any boundary orbit (see Definition 3.1) of $\Gamma(\mathrm{mod}\ kQ)$ contains at most 2 modules.

For the representation-infinite case, see [5, 7, 8].
As a consequence, the answer to Problem 1.1(1) is given in the following theorem.

Theorem 1.3. *Let Q be a finite connected acyclic quiver.*
(1) [[7], Theorem 3.1] *If Q is a non-Dynkin quiver, then the poset of basic pre-projective tilting kQ-modules is a distributive lattice if and only if the degree of all vertices in Q are greater than* 1.
(2) If Q is a Dynkin quiver, then the poset of basic tilting kQ-modules is a distributive lattice if and only if Q is of types $\mathbb{A}_1, \mathbb{A}_2$ or $\mathbb{A}_3$ with a nonlinear orientation.

On the other hand, we also show the following result which answers Problem 1.1(2).

Theorem 1.4. *Let Q be a finite connected acyclic quiver. Then the poset of basic support τ-tilting kQ-modules is a distributive lattice if and only if Q is of type $\mathbb{A}_1$.*

The paper is organized as follows. In section 2 we recall some preliminary definitions and results of tilting theory, τ-tilting theory and lattice theory, especially about the tilting quiver, support τ-tilting quiver and distributive lattice. In subsection 3.1 we first introduce the notions of boundary module and boundary orbit and then prove Theorem 1.2. In subsection 3.2 we give a proof of Theorem 1.3 by using Theorem 1.2. In subsection 3.3 we prove Theorem 1.4.

2. Preliminaries

2.1. **Tilting theory and τ-tilting theory.** We start with the following definitions of tilting modules and tilting quiver which was considered in [7], and was first introduced in [4, 13].

Definition 2.1. *A module $T \in \mathrm{mod}\text{-}kQ$ is a* tilting module *if*
(1) $\mathrm{Ext}^1_{kQ}(T, T) = 0$.
(2) $|T| = |Q_0|$.

We denote by $\mathcal{T}(Q)$ a complete set of representatives of the isomorphism classes of the basic tilting modules in mod-kQ.

Definition 2.2. *The* tilting quiver $\vec{\mathcal{T}}(Q)$ *is defined as follows:*
(1) $\vec{\mathcal{T}}(Q)_0 := \mathcal{T}(Q)$.
(2) $T \to T'$ *in* $\vec{\mathcal{T}}(Q)$ *if* $T \cong M \oplus X$, $T' \cong M \oplus Y$ *for some* $X, Y \in \mathrm{ind}\text{-}kQ$, $M \in \mathrm{mod}\text{-}kQ$ *and there is a non-split exact sequence*

$$0 \longrightarrow X \longrightarrow M' \longrightarrow Y \longrightarrow 0$$

with $M' \in \mathrm{add}\ M$.

Now we recall some basic definitions of τ-tilting theory, which was first introduced in [2], in order to “complete” the classical tilting theory from the viewpoint of mutation.

Definition 2.3. *(1) We call* $M \in \text{mod-}kQ$ τ-rigid *if* $\mathrm{Hom}_{kQ}(M, \tau M) = 0$.

(2) We call $M \in \text{mod-}kQ$ τ-tilting *if M is τ-rigid and* $|M| = |Q_0|$.

(3) We call $M \in \text{mod-}kQ$ support τ-tilting *if there exists an idempotent e of kQ such that M is a τ-tilting* $(kQ/\langle e\rangle)$*-module.*

We denote by $\mathcal{ST}(Q)$ a complete set of representatives of the isomorphism classes of the basic support τ-tilting modules in mod-kQ.

Recall that the Hasse-quiver $\vec{P}$ of a poset $(P, \leq)$ is defined as follows:

(1) $\vec{P}_0 := P$.

(2) $x \to y$ in $\vec{P}$ if $x > y$ and there is no $z \in P$ such that $x > z > y$.

The support τ-tilting quiver $\overrightarrow{\mathcal{ST}}(Q)$ is defined as follows.

Proposition-Definition 2.1 ([2], Theorem 2.7, Corollary 2.34)**.** *(1) Let* $T, T' \in \mathcal{ST}(Q)$*, then the following relation* $\leq$ *defines a partial order on* $\mathcal{ST}(Q)$*,*

$$T \geq T' \overset{\text{def}}{\Leftrightarrow} \mathrm{Fac}T \supseteq \mathrm{Fac}T'$$

.

(2) The support τ-tilting quiver $\overrightarrow{\mathcal{ST}}(Q)$ *is the Hasse quiver of the partial order set* $(\mathcal{ST}(Q), \leq)$*.*

We remark that there is the following similar result in the classical tilting theory.

Theorem 2.4 ([4], Theorem 2.1)**.** *(1) Let* $T, T' \in \mathcal{T}(Q)$*, then the following relation* $\leq$ *defines a partial order on* $\mathcal{T}(Q)$*,*

$$T \geq T' \overset{\text{def}}{\Leftrightarrow} \mathrm{Fac}T \supseteq \mathrm{Fac}T'$$

.

(2) The tilting quiver $\overrightarrow{\mathcal{T}}(Q)$ *is the Hasse quiver of the partial order set* $(\mathcal{T}(Q), \leq)$*.*

We end this subsection with the following two examples.

Example 2.1. *Let* Q_1, Q_2 *be the following two different quivers, see Figure 1. Although they share the same underlying graph, however, the corresponding tilting quivers are different.*

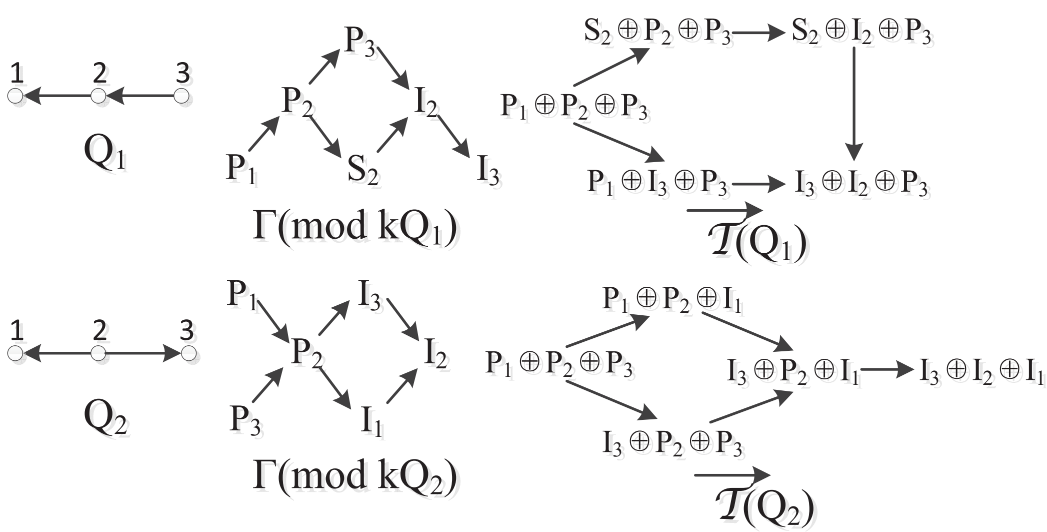

Figure 1

Example 2.2. *Let Q be of type* $\mathbb{A}_2$*, then its support τ-tilting quiver* $\overrightarrow{\mathcal{ST}}(Q)$ *is shown in Figure 2.*

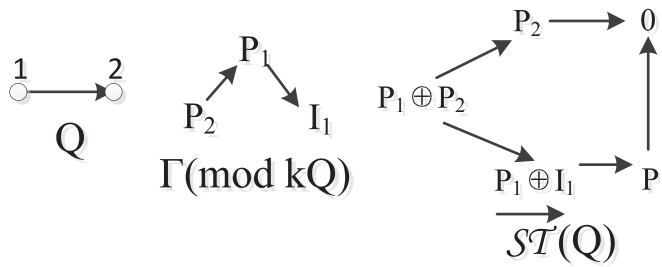

Figure 2

2.2. **Lattices and distributive lattices.** In this subsection we will recall definitions of lattices and distributive lattices.

Definition 2.5. *A poset* $(L, \leq)$ *is a* lattice *if for any* $x, y \in L$ *there is a minimum element of* $\{z \in L | z \geq x, y\}$ *and there is a maximum element of* $\{z \in L | z \leq x, y\}$.

In this case, we denote by $x \vee y$ *the minimum element of* $\{z \in L | z \geq x, y\}$ *and call it* join *of* x *and* y. *We also denote by* $x \wedge y$ *the maximum element of* $\{z \in L | z \leq x, y\}$ *and call it* meet *of* x *and* y.

Definition 2.6. *A lattice* L *is a* distributive lattice *if* $(x \vee y) \wedge z = (x \wedge z) \vee (y \wedge z)$ *holds for any* $x, y, z \in L$.

Immediately we have the following basic observation, which will be used frequently in this paper.

Lemma 2.7. *For any* $n \geq 2$, *the following Hasse quiver in Figure 3 is not a distributive lattice.*

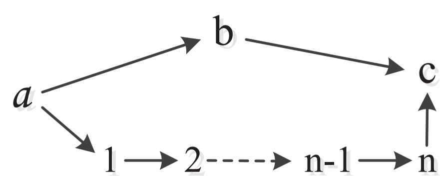

Figure 3

Proof. Since $n \geq 2$, it is easy to see that

$$(b \vee 2) \wedge 1 = a \wedge 1 = 1 \neq 2 = c \vee 2 = (b \wedge 1) \vee (2 \wedge 1),$$

therefore it is not a distributive lattice. □

In the above examples 2.1 and 2.2, it is easy to see that the lattice $(\mathcal{T}(Q_2), \leq)$ is a distributive lattice. On the other hand, it follows by Lemma 2.7 that both $(\mathcal{T}(Q_1), \leq)$ and $(\overrightarrow{\mathcal{ST}}(Q), \leq)$ are not distributive lattice.

3. Main results

3.1. **Boundary module and boundary orbit.** From now on, we will not distinguish between an indecomposable kQ-module M and its corresponding vertex $[M]$ in the Auslander-Reiten quiver $\Gamma(\text{mod } kQ)$. We will also not distinguish between a poset $(P, \leq)$ and its Hasse quiver $\vec{P}$.

By Theorem 2.4 and Proposition-Definition 2.1, it is easy to see that our problem reduces to the study of lattice structure of the tilting quiver $\vec{\mathcal{T}}(Q)$ and the support τ-tiling quiver $\overrightarrow{\mathcal{ST}}(Q)$.

Before proceeding further, let (Γ, τ) be a connected translation quiver, recall from [1] that a connected full subquiver Σ of Γ is called a *presection* (is also called a *cut* in [10]) in Γ if it satisfies the following two conditions:

(1) If $x \in \Sigma_0$ and $x \to y$ is an arrow, then either $y \in \Sigma_0$ or $\tau y \in \Sigma_0$.

(2) If $y \in \Sigma_0$ and $x \to y$ is an arrow, then either $x \in \Sigma_0$ or $\tau^{-1}x \in \Sigma_0$.

Moreover, in [9] a connected full subquiver Σ of Γ is a called *section* of Γ if the following conditions are satisfied:

(1) Σ contains no oriented cycle.

(2) Σ meets each τ-orbit in Γ exactly once.

(3) Σ is convex in Γ, that is, every path in Γ with end-points belonging to Σ lies entirely in Σ.

From [11] recall also that a module S is said to be a *slice module* if S is sincere and add S satisfies the following conditions:

(1) If there is a path $x_0 \to x_1 \to \cdots \to x_t$ with $x_0, x_t \in \operatorname{add} S$ in the Auslander-Reiten quiver, then $x_i \in \operatorname{add} S$ $(i = 0, 1, \cdots, t)$.

(2) If M is indecomposable and not projective, then at most one of $M, \tau M$ belongs to add S.

(3) If there is an arrow $M \to X$ with $X \in \operatorname{add} S$ in the Auslander-Reiten quiver, then either $M \in \operatorname{add} S$ or M is not injective and $\tau^{-1}M \in \operatorname{add} S$.

Now we introduce the notions of boundary module and boundary orbit.

Definition 3.1. *(1) We call a module $M \in \Gamma(\mathrm{mod}\ kQ)$* boundary module *if M has at most one direct predecessor and at most one direct successor in Auslander-Reiten quiver $\Gamma(\mathrm{mod}\ kQ)$.*

(2) We call a τ-orbit Σ of $\Gamma(\mathrm{mod}\ kQ)$ boundary orbit *if Σ contains a boundary module.*

The following observation is useful.

Lemma 3.2. *Let Q be a Dynkin quiver. If one of its boundary orbits contains at least 3 modules, then the tilting quiver $\vec{\mathcal{T}}(Q)$ is not a distributive lattice.*

Proof. Since Q is a Dynkin quiver, $\Gamma(\mathrm{mod}\ kQ)$ must be a full convex subquiver of $\mathbb{Z}Q$. Without loss of generality, by our assumption $\Gamma(\mathrm{mod}\ kQ)$ will contain the following shaded area $\mathcal{T}$, see Figure 4.

Now we enlarge $\mathcal{T}$ for each type, for the type A, see the left-lower of Figure 4. For simplicity, we may continue with the type A, for the remaining two types, the argument is similar.

Let $|Q_0| = n$, it is easy to see that we can construct a section Σ of the lower $(n-2)$-rows starting with M_6 and denote the module corresponding to this section by M_Σ. Then we consider the following five modules

$$T_1 = M_\Sigma \oplus M_4 \oplus M_1, T_2 = M_\Sigma \oplus M_4 \oplus M_2, T_3 = M_\Sigma \oplus M_5 \oplus M_2,$$
$$T_4 = M_\Sigma \oplus M_5 \oplus M_3, T_5 = M_\Sigma \oplus M_1 \oplus M_3.$$

Since $\Gamma(\mathrm{mod}\ kQ)$ is a standard component, it is not hard to see that all of these five modules are tilting modules and they forms the right-lower of Figure 4, which is a full subquiver of the tilting quiver $\vec{\mathcal{T}}(Q)$, however, is not a distributive lattice by Lemma 2.7. Hence the tilting quiver $\vec{\mathcal{T}}(Q)$ is also not a distributive lattice, which completes the proof. □

Now we are ready to prove Theorem 1.2.

(1) $\Leftrightarrow$ (2): This is shown in [14] or [15].

(2) $\Rightarrow$ (3): Let $|Q_0| = n$, according to (2) it follows that any tilting module can be written as

$$T \cong \bigoplus_{i=1}^{n} \tau^{-r_i} P_i$$

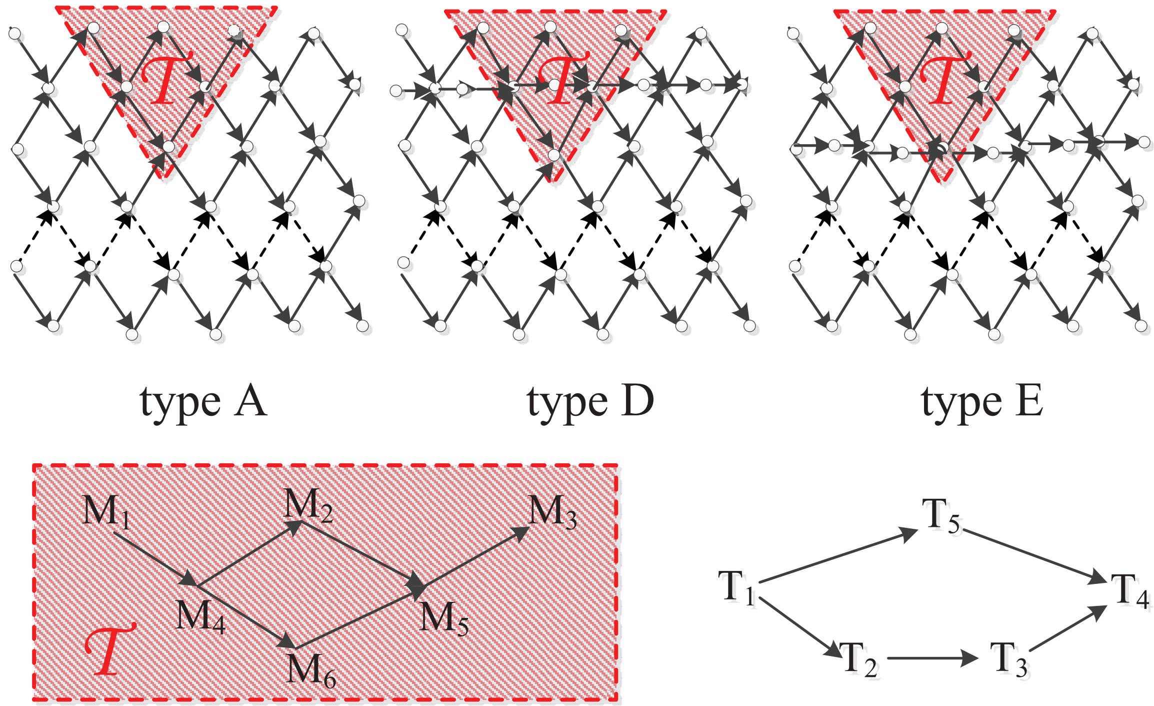

Figure 4

for $r_i \in \mathbb{Z}_{\geq 0}, 1 \leq i \leq n$ and if T, T' be two tilting modules, $T \to T'$ in $\vec{\mathfrak{T}}(Q)$ if and only if there is an indecomposable direct summand X such that $T \cong M \oplus X$ and $T' \cong M \oplus \tau^{-1}X$. Thus, for any two tilting modules $T \cong \bigoplus_{i=1}^{n} \tau^{-r_i}P_i, T' \cong \bigoplus_{i=1}^{n} \tau^{-r'_i}P_i$, $T \geq T'$ if and only if $r_i \leq r'_i, 1 \leq i \leq n$.

From now on let Σ_T be the full subquiver of $\Gamma(\text{mod } kQ)$ generated by T. Since $\Sigma_T, \Sigma_{T'}$ form a section of $\Gamma(\text{mod } kQ)$, it is not hard to check that both $\Sigma_{\bigoplus_{i=1}^{n} \tau^{-\min\{r_i, r'_i\}}P_i}$ and $\Sigma_{\bigoplus_{i=1}^{n} \tau^{-\max\{r_i, r'_i\}}P_i}$ again form a section of $\Gamma(\text{mod } kQ)$, which implies that both $\bigoplus_{i=1}^{n} \tau^{-\min\{r_i, r'_i\}}P_i$ and $\bigoplus_{i=1}^{n} \tau^{-\max\{r_i, r'_i\}}P_i$ are tilting modules. Therefore the join and meet of T and T' are

$$T \vee T' \cong \bigoplus_{i=1}^{n} \tau^{-\min\{r_i, r'_i\}}P_i, \ T \wedge T' \cong \bigoplus_{i=1}^{n} \tau^{-\max\{r_i, r'_i\}}P_i$$

respectively, which makes the tilting quiver $\vec{\mathfrak{T}}(Q)$ to be a distributive lattice. Indeed, it follows by the fact that $a \vee b = (\min(r_i, r'_i))_{1 \leq i \leq n}$ and $a \wedge b = (\max(r_i, r'_i))_{1 \leq i \leq n}$ makes $(\mathbb{Z}^n, \leq^{\text{op}})$ to be a distributive lattice, where $a = (r_i)_{1 \leq i \leq n}, b = (r'_i)_{1 \leq i \leq n}$.

(3) $\Rightarrow$ (4): It follows from Lemma 3.2 at once.

(4) $\Rightarrow$ (2): Since any boundary orbit of $\Gamma(\text{mod } kQ)$ contains at most 2 modules and $\Gamma(\text{mod } kQ)$ is a full convex subquiver of $\mathbb{Z}Q$, it follows that $\Gamma(\text{mod } kQ)$ is bounded by the following shaded area $\mathcal{R}$, see Figure 5.

Since $\Gamma(\text{mod } kQ)$ is a standard component, we have that for any $M, N \in \mathcal{R}$, if there exists a path from M to τN, then $\text{Hom}_{kQ}(M, \tau N) \neq 0$.

Let T be any tilting module, because $\text{Ext}^1_{kQ}(T, T) = \text{Hom}_{kQ}(T, \tau T) = 0$, so there is no path from T_i to τT_j, which implies that Σ_T meets each τ-orbit at most once. Moreover, since $|(\Sigma_T)_0| = |T| = |Q_0|$, it follows that Σ_T meets each τ-orbit exactly once.

According to [[1], Proposition 1.7], it suffices to prove that Σ_T is a presection of $\Gamma(\text{mod } kQ)$. Indeed, if $x \in (\Sigma_T)_0$, $x \to y$ is an arrow and $y, \tau y \notin (\Sigma_T)_0$, then there exists $i \neq 0, 1$ such that

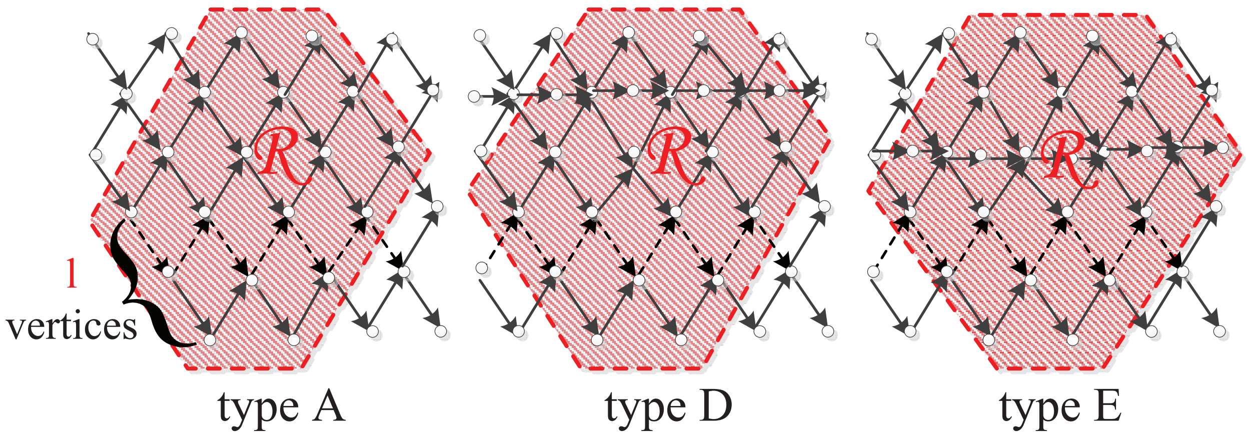

Figure 5

$\tau^i y \in (\Sigma_T)_0$. If $i \geq 2$, then there exists a path from $\tau^i y$ to τx, $\tau^i y, x \in (\Sigma_T)_0 \subseteq \Gamma(\text{mod } kQ) \subseteq \mathcal{R}$, thus we have $\text{Hom}_{kQ}(\tau^i y, \tau x) \cong \text{Ext}^1_{kQ}(x, \tau^i y) \neq 0$, which contradicts that $\tau^i y, x \in (\Sigma_T)_0$ and T is a tilting module. For the $i \leq -1$ case, the proof is similar.

Using the same argument as above, we can easily carry out that if $y \in (\Sigma_T)_0$, $x \to y$ is an arrow, then either $x \in (\Sigma_T)_0$ or $\tau^{-1} x \in (\Sigma_T)_0$. Finally the connectivity of Σ_T follows from the connectivity of $\Gamma(\text{mod } kQ)$, which completes the proof.

3.2. **Proof of Theorem 1.3.** In this subsection we start to prove Theorem 1.3.

For the non-Dynkin case, see [[7], Theorem 3.1]. If Q is a Dynkin quiver, we divide into the following three cases.

Case 1: Q **is of type** A**.**

$|Q_0| = 1, 2$, then the tilting quivers are $\cdot, \cdot \to \cdot$, respectively, it is clear.

$|Q_0| = 3$, see Example 2.1 and it is easy to see that the tilting quiver of $\cdot \to \cdot \leftarrow \cdot$ is also a distributive lattice.

$|Q_0| = 4$, then we can list all the non-isomorphic quivers and their corresponding Auslander-Reiten quivers as follows, see Figure 6.

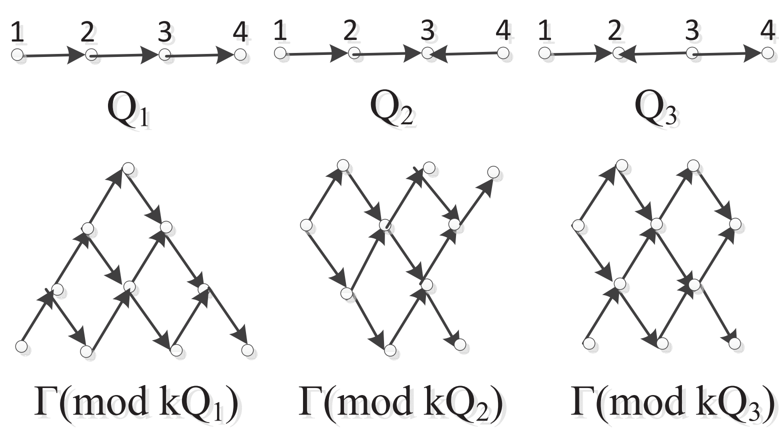

Figure 6

Since for each of these three Auslander-Reiten quivers, we can always find a boundary orbit containing 3 modules, then by Theorem 1.2 the corresponding tilting quiver $\vec{\mathcal{T}}(Q_i)$ is not a distributive lattice, $1 \leq i \leq 3$.

$|Q_0| \geq 5$, if the tilting quiver $\vec{\mathcal{T}}(Q)$ is a distributive lattice, then by Theorem 1.2 any boundary orbit of $\Gamma(\text{mod } kQ)$ contains at most 2 modules, i.e., $\Gamma(\text{mod } kQ)$ is bounded by the shaded area $\mathcal{R}$ of Figure 5.

Let $|Q_0| = n \geq 5$ and l be defined in Figure 5. It is well known that the number of indecomposable kQ-modules is $\frac{n(n+1)}{2}$. On the other hand, there are at most $l(n+1-l)+n$ modules in $\mathcal{R}$, $1 \leq l \leq n$. However, when $n \geq 5$ we have

$$l(n+1-l)+n = -(l-\frac{n+1}{2})^2 + \frac{n^2+6n+1}{4} \leq \frac{n^2+6n+1}{4} < \frac{n(n+1)}{2}$$

which contradicts that $\Gamma(\text{mod } kQ)$ is bounded by $\mathcal{R}$.

Case 2: Q **is of type** D**.**

Similarly, if the tilting quiver $\vec{\mathcal{T}}(Q)$ is a distributive lattice, then $\Gamma(\text{mod } kQ)$ is bounded by $\mathcal{R}$. Let $|Q_0| = n \geq 4$ and l is defined in the same way, then on one hand the number of indecomposable kQ-modules is $n(n-1)$; On the other hand, there are at most $l(n-l)+n+3$ modules in $\mathcal{R}$, $1 \leq l \leq n-1$. However, when $n \geq 4$ we have

$$l(n-l)+n+3 = -(l-\frac{n}{2})^2 + \frac{n^2+4n+12}{4} \leq \frac{n^2+4n+12}{4} < n(n-1)$$

the same contradiction follows.

Case 3: Q **is of type** E**.**

We now proceed as in the proof of above two cases. On one hand, when $n = 6, 7, 8$, the number of indecomposable kQ-modules is $36, 63, 120$, respectively. On the other hand, there are at most $l(n-l)+n+4$ modules in $\mathcal{R}$, $1 \leq l \leq n-1$. However,

$$l(n-l)+n+4 = -(l-\frac{n}{2})^2 + \frac{n^2+4n+16}{4} \leq \frac{n^2+4n+16}{4}$$

which equals to $19, 23.25, 28$ respectively when $n = 6, 7, 8$, now we have the same contradiction.

Finally, by combining the above three cases together, we complete the proof of Theorem 1.3(2).

3.3. **Proof of Theorem 1.4.** Indeed, by [[6], Theorem 0.4] it suffices to consider the following two cases.

Case 1: Q **is of Dynkin type.**

If $|Q_0| = 1$, then the support τ-tilting quiver is $\cdot \to \cdot$, it is clear.

If $|Q_0| = n \geq 2$, then Q contains $\mathbb{A}_2$ as its full subquiver. Without loss of generality we assume that $\{e_1, \cdots, e_n\}$ is a complete set of primitive orthogonal idempotents for kQ and there is an arrow α between the vertices 1 and 2. Let $e = e_3 + e_4 + \cdots + e_n$, then $kQ/\langle e \rangle \cong k\mathbb{A}_2$.

By Example 2.2 the support τ-tilting quiver $\mathcal{S}\vec{\mathcal{T}}(\mathbb{A}_2)$ is not a distributive lattice. On the other hand, according to [[2], Proposition 2.27(a)] it can easily be seen that $\mathcal{S}\vec{\mathcal{T}}(\mathbb{A}_2)$ is a full subquiver of $\mathcal{S}\vec{\mathcal{T}}(Q)$, which implies that $\mathcal{S}\vec{\mathcal{T}}(Q)$ is not a distributive lattice itself.

Case 2: Q **has at most** 2 **vertices.**

According to [[6], Proposition 2.2], it follows that the support τ-tilting quiver $\mathcal{S}\vec{\mathcal{T}}(Q)$ is isomorphic to the Figure 3 in Lemma 2.7, where n tends to $+\infty$. Now by Lemma 2.7 it is obvious that $\mathcal{S}\vec{\mathcal{T}}(Q)$ is not a distributive lattice.

Finally, by combining the above two cases together, we complete the proof of Theorem 1.4.

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